

ABELIAN VARIETIES ISOGENOUS TO A POWER OF AN ELLIPTIC CURVE

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ABSTRACT. Let E be an elliptic curve over a field k . Let $R := \text{End } E$. There is a functor $\mathcal{H}om_R(-, E)$ from the category of finitely presented torsion-free left R -modules to the category of abelian varieties isogenous to a power of E , and a functor $\text{Hom}(-, E)$ in the opposite direction. We prove necessary and sufficient conditions on E for these functors to be equivalences of categories.

1. INTRODUCTION

Let E be an elliptic curve over a field k . Let $R := \text{End } E$. We would like to classify all abelian varieties isogenous to a power of E .

Serre [21, pp. 50–51] defined a functor $\mathcal{H}om_R(-, E)$ that takes as input a finitely presented (f.p.) left R -module M and produces a commutative group scheme. We will prove that when restricted to torsion-free modules, this becomes a fully faithful functor of additive categories

$$\begin{aligned} \mathcal{H}om_R(-, E): \{\text{f.p. torsion-free left } R\text{-modules}\}^{\text{opp}} \\ \rightarrow \{\text{abelian varieties isogenous to a power of } E\}. \end{aligned} \quad (1)$$

In the other direction, we have a functor

$$\begin{aligned} \text{Hom}(-, E): \{\text{abelian varieties isogenous to a power of } E\} \\ \rightarrow \{\text{f.p. torsion-free left } R\text{-modules}\}^{\text{opp}} \end{aligned} \quad (2)$$

that provides the inverse on the essential image of (1). This is useful because the modules can be classified for each possible R .

We find necessary and sufficient conditions on E for (1) and (2) to be equivalences of categories. For simplicity, in this introduction we state the answer only for elliptic curves over finite fields.

Theorem 1.1. *Let E be an elliptic curve over a finite field $k = \mathbb{F}_q$. Let $R := \text{End } E$. Let $\pi \in R$ be the q -power Frobenius endomorphism. Then (1) and (2) are equivalences of categories if and only if one of the following holds:*

- E is ordinary and $\mathbb{Z}[\pi] = R$;
- E is supersingular, $k = \mathbb{F}_p$, and $\mathbb{Z}[\pi] = R$; or
- E is supersingular, $k = \mathbb{F}_{p^2}$, and R is of rank 4 over \mathbb{Z} .

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Theorem 1.1 is close to many results in the literature. Waterhouse in [25] proves many results relating the isogeny class of an elliptic curve E to the ideal classes of $\text{End } E$, and he also considers such issues when E is replaced by an abelian variety. An analogue of Theorem 1.1 with the functors $\mathcal{H}om$ and $\text{Hom}(-, E)$ replaced by a similarly-defined functor \otimes and $\text{Hom}(E, -)$ is proved in Serre's appendix to [11] in the case where R is the maximal order in an imaginary quadratic field. Other cases are handled in [23], [10], [20], and especially Kani's work [9]; although these works do not define the functor $\mathcal{H}om$, they too classify all abelian varieties isogenous to a power of E in the case where E is ordinary and $\text{rk } R = 2$ (see Theorems 1, 2, and 3 of [9]). In fact, at one point (in the proof of our Theorem 4.6(a)), we make use of one of the easier results of [9].

The category of all ordinary abelian varieties over a finite field is equivalent to the category of *Deligne modules*, which are finitely-generated torsion-free \mathbb{Z} -modules provided with an endomorphism that corresponds to the Frobenius; see [5] for the equivalence of categories, and [8] for the translation of polarizations and other geometric terms into the category of Deligne modules. The ordinary case of Theorem 1.1 is an easy consequence of Deligne's more general equivalence of categories. We include this case here because it also follows naturally from the work we had to do to study non-ordinary elliptic curves and elliptic curves over infinite fields.

Let us now outline the rest of the paper. Section 2 introduces notation to be used. If R is the endomorphism ring of an elliptic curve, then R is \mathbb{Z} , an imaginary quadratic order, or a maximal quaternionic order; Section 3 reviews the classification of f.p. torsion-free left R -modules in each case, and in a little more generality. Section 4 introduces the two functors above and proves their basic properties; in particular it is shown that applying $\mathcal{H}om_R(-, E)$ to torsion-free modules produces abelian varieties isogenous to a power of E . Moreover, Section 4.3 relates duality of modules to duality of abelian varieties. Section 5 proves that when E is a supersingular elliptic curve over \mathbb{F}_{p^2} with $\text{rk } \text{End } E = 4$, the functors (1) and (2) are equivalences of categories, so that there is a clean classification of abelian varieties isogenous to a power of E . In preparation for the other cases, Section 6 defines the notion of a kernel subgroup, and shows that the functors (1) and (2) are equivalences of categories if and only if every finite subgroup scheme of every power of E is a kernel subgroup. All this is combined in Section 7, which gives a complete answer to the question of when (1) and (2) are equivalences of categories.

2. NOTATION

Let R be a noetherian integral domain. Let $K = \text{Frac } R$. The *torsion submodule* of an R -module M is

$$M_{\text{tors}} := \{m \in M : rm = 0 \text{ for some nonzero } r \in R\}.$$

Call M *torsion-free* if $M_{\text{tors}} = 0$. Call a submodule N of M (or an injection $N \rightarrow M$) *saturated* if the cokernel of $N \rightarrow M$ is torsion-free. Given a f.p. R -module M , define its *rank* as $\text{rk } M := \dim_K(K \otimes_R M)$. The notion of rank extends to f.p. left modules over a subring R in a division algebra K .

If k is a field, let \bar{k} be an algebraic closure of k , let k_s be the separable closure of k in \bar{k} , and let $\mathcal{G}_k := \text{Gal}(k_s/k)$. If G is a finite group scheme over a field k , its *order* is $\#G := \dim_k \Gamma(G, \mathcal{O}_G)$. If A is any commutative group scheme over a field k , and $n \in \mathbb{Z}_{>0}$,

then $A[n]$ denotes the group scheme kernel of $A \xrightarrow{n} A$. If ℓ is a prime not equal to $\text{char } k$, then the ℓ -adic *Tate module* of A is

$$T_\ell A := \varprojlim_e A[\ell^e](k_s).$$

If X is a scheme over a field k of characteristic $p > 0$, and q is a power of p , let $\pi_{X,q}: X \rightarrow X^{(q)}$ be the q -power Frobenius morphism; if $k = \mathbb{F}_q$, then let π_X be $\pi_{X,q}: X \rightarrow X$. If E is a commutative group scheme over a field k , then $\text{End } E$ denotes its endomorphism ring as a commutative group scheme over k , i.e., the ring of endomorphisms defined over k ; the same comment applies to Hom .

Recall that the *essential image* of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of the objects of \mathcal{D} isomorphic to FC for some $C \in \mathcal{C}$; from now on, we call this simply the *image* of F .

3. CLASSIFYING TORSION-FREE MODULES

3.1. Dedekind domains. Suppose that R is a Dedekind domain. Finitely presented (henceforth denoted f.p.) torsion-free R -modules can be completely classified, as is well known [18, Theorem 4.13]. To describe the result, we need the notion of determinant of a module. Given a torsion-free R -module M of rank r , its *determinant* $\det M := \bigwedge^r M$ is a f.p. torsion-free R -module of rank 1; sometimes we identify $\det M$ with its class in $\text{Pic } R$. For example, if $M = I_1 \oplus \cdots \oplus I_r$, where each I_j is a nonzero ideal of R , then $\text{rk } M = r$ and

$$\det M \simeq I_1 \otimes_R \cdots \otimes_R I_r \simeq I_1 \cdots I_r \quad (\text{the product ideal in } R).$$

Theorem 3.1.

- (a) *A f.p. R -module is torsion-free if and only if it is projective.*
- (b) *Every f.p. projective R -module is isomorphic to a finite direct sum of invertible ideals.*
- (c) *The isomorphism type of a f.p. projective R -module is determined by its rank and determinant.*
- (d) *Every pair $(r, c) \in \mathbb{Z}_{>0} \times \text{Pic } R$ arises as the rank and determinant of a nonzero f.p. projective R -module M ; one representative is $M := R^{r-1} \oplus I$ where $[I] = c$.*

3.2. Quadratic orders. For a general order in a Dedekind domain, the structure theory of torsion-free f.p. modules is wild. Fortunately, for *quadratic* orders there is a theory that is only slightly more complicated than that for Dedekind domains. Recall that if R_{\max} is the ring of integers in a quadratic field K , then every order in K is of the form $R_f := \mathbb{Z} + fR_{\max}$ for a positive integer f called the *conductor*. The orders containing R_f are the orders R_g for $g|f$.

Theorem 3.2. *Let R be a quadratic order, i.e., an order in a degree 2 extension K of \mathbb{Q} . Let M be a f.p. torsion-free R -module.*

- (i) *There exists a unique chain of orders $R_1 \subseteq \cdots \subseteq R_n$ between R and K and invertible ideals I_1, \dots, I_n of R_1, \dots, R_n , respectively, such that $M \simeq I_1 \oplus \cdots \oplus I_n$ as an R -module.*
- (ii) *The I_i are not unique, but their product $I_1 \cdots I_n$ is an invertible R_n -ideal whose class $[M] \in \text{Pic } R_n$ depends only on M .*
- (iii) *The isomorphism type of M is uniquely determined by the chain $R_1 \subseteq \cdots \subseteq R_n$ and the class $[M] \in \text{Pic } R_n$.*

Proof. See [3]. For generalizations to other integral domains, see [2, Section 7], [4], [12], and the survey article [19]. \square

3.3. Maximal orders in quaternion algebras. Let B be a quaternion division algebra over \mathbb{Q} . Let \mathcal{O} be a maximal order in B . Given a f.p. left \mathcal{O} -module M , the nonnegative integer $\text{rk } M$ is the dimension of the left B -vector space $B \otimes_{\mathcal{O}} M$, which is also $\frac{1}{4} \text{rk}_{\mathbb{Z}} M$. Call M *torsion-free* if the natural map $M \rightarrow B \otimes_{\mathcal{O}} M$ is an injection, or equivalently if M is torsion-free as a \mathbb{Z} -module.

The classification of f.p. torsion-free left \mathcal{O} -modules is similar to the classification over a Dedekind domain, and even simpler in ranks at least 2.

Theorem 3.3.

- (a) *A f.p. left \mathcal{O} -module is torsion-free if and only if it is projective.*
- (b) *Every f.p. projective left \mathcal{O} -module is isomorphic to a finite direct sum of ideals.*
- (c) *A f.p. projective left \mathcal{O} -module of rank at least 2 is free.*

Proof.

- (a) See [18, Corollary 21.5].
- (b) This follows from the final statement of [18, Corollary 21.5].
- (c) This is a classical result due to Eichler [6]; see also [22, Theorem 3.5]. \square

4. CATEGORICAL CONSTRUCTIONS

4.1. A functor to an abelian category. We recall the following general construction (cf. [21, pp. 50–51] and [25, Appendix]). Given an abelian category \mathcal{C} , an object $E \in \mathcal{C}$, a ring R , and a ring homomorphism $R \rightarrow \text{End } E$, there is a left exact functor

$$\mathcal{H}om_R(-, E): \{\text{f.p. left } R\text{-modules}\}^{\text{opp}} \rightarrow \mathcal{C}. \quad (3)$$

Given a f.p. left R -module M , the object $\mathcal{H}om_R(M, E)$ can be characterized in two ways:

- It represents the functor $\text{Hom}_R(M, \text{Hom}(-, E)): \mathcal{C} \rightarrow \mathbf{Sets}$.
- Explicitly, if M is the cokernel of a homomorphism $R^m \rightarrow R^n$, then $\mathcal{H}om_R(M, E)$ is the kernel of the corresponding morphism $E^n \rightarrow E^m$.

The first characterization implies that for any object $F \in \mathcal{C}$,

$$\text{Hom}(F, \mathcal{H}om_R(M, E)) = \text{Hom}_R(M, \text{Hom}(F, E)); \quad (4)$$

cf. [21, p. 52].

4.2. The functor for an elliptic curve produces abelian varieties. The category of commutative proper group schemes over a field k is an abelian category (the hardest part of this statement is the existence of cokernels, which is [7, Corollaire 7.4]). From now on, we assume that \mathcal{C} is this category.

Proposition 4.1. *Let E be an abelian variety over a field k . Let R be a domain that is f.p. as a \mathbb{Z} -module. Let $R \rightarrow \text{End } E$ be a ring homomorphism. Let M be a f.p. left R -module. Let $A := \mathcal{H}om_R(M, E)$. Then $\dim A = (\text{rk } M)(\dim E)$.*

Proof. For any $n \geq 1$, we have $\mathcal{H}om_R(R/nR, E) = E[n]$, by the second description of $\mathcal{H}om_R(-, E)$. If M is torsion, then it is a quotient of $(R/nR)^m$ for some $m, n \geq 1$; then $A \subseteq E[n]^m$, so A is finite.

In general, let $r = \text{rk } M$. There is an exact sequence

$$0 \rightarrow R^r \rightarrow M \rightarrow T \rightarrow 0$$

for some torsion module T ; this yields

$$0 \rightarrow \mathcal{H}om_R(T, E) \rightarrow A \rightarrow E^r. \quad (5)$$

By the previous paragraph, $\mathcal{H}om_R(T, E)$ is finite, so $\dim A \leq r \dim E$. There exists a nonzero $\rho \in R$ such that $\rho T = 0$. Since R is f.p. as a \mathbb{Z} -module, it follows that there exists a positive integer n such that $nT = 0$. Then $R^r \xrightarrow{n} R^r$ factors as $R^r \hookrightarrow M \rightarrow R^r$, which induces $E^r \rightarrow A \rightarrow E^r$ whose composition is multiplication by n , which is surjective. Thus $A \rightarrow E^r$ is surjective, so $\dim A \geq r \dim E$. Hence $\dim A = r \dim E$. \square

If E is an elliptic curve, and I is a subset of $\text{End } E$, let $E[I] := \bigcap_{\alpha \in I} \ker \alpha$.

Theorem 4.2. *Let E be an elliptic curve over a field k . Let R be a saturated subring of $\text{End } E$ (saturated as a \mathbb{Z} -module). Let M be a torsion-free f.p. left R -module. Let $A := \mathcal{H}om_R(M, E)$. Then*

- (a) *The group scheme A is an abelian variety isogenous to a power of E .*
- (b) *The functor $\mathcal{H}om_R(-, E)$ is exact.*
- (c) *If $f: E^r \rightarrow E^s$ is a homomorphism arising from applying $\mathcal{H}om_R(-, E)$ to an R -homomorphism $g: R^s \rightarrow R^r$, then the image of f is isomorphic to $\mathcal{H}om_R(N, E)$ for some f.p. torsion-free R -module $N \subseteq R^r$. (Moreover, if $R = \text{End } E$, then every homomorphism $f: E^r \rightarrow E^s$ arises from some g .)*
- (d) *If I is a nonzero left R -ideal, then $\mathcal{H}om_R(R/I, E) \simeq E[I]$ and $\mathcal{H}om_R(I, E) \simeq E/E[I]$.*
- (e) *If T is an R -module that is finite as a set, then $\mathcal{H}om_R(T, E)$ is a finite group scheme of order $(\#T)^{2/\text{rk } R}$.*
- (f) *If $n \in \mathbb{Z}_{>0}$, then $A[n] \simeq \mathcal{H}om_R(M, E[n])$, where the latter is defined by using the induced ring homomorphism $R \rightarrow \text{End } E[n]$.*
- (g) *If ℓ is a prime not equal to $\text{char } k$, then $T_\ell A \simeq \text{Hom}_R(M, T_\ell E)$.*

Proof.

- (a) Let $r = \text{rk } M = \dim A$. The proof of Proposition 4.1 shows that A admits a surjection to E^r with finite kernel, so if A is an abelian variety, it is isogenous to E^r .

The ring R is either \mathbb{Z} , a quadratic order, or a maximal quaternionic order. In the first and third cases, M is projective of rank r over R (the quaternionic case is Theorem 3.3(a)); in other words, M is a direct summand of R^n for some n ; thus A is a direct factor of E^n , so A is an abelian variety.

So suppose that R is a quadratic order. Let c be the conductor, i.e., the index of R in its integral closure. Let ℓ denote a prime. If $\ell \nmid c$, then the semi-local ring $R \otimes \mathbb{Z}_{(\ell)}$ is a Dedekind domain, but a semi-local Dedekind domain is a principal ideal domain, so $M \otimes \mathbb{Z}_{(\ell)}$ is free of rank r over $R \otimes \mathbb{Z}_{(\ell)}$, and $M/\ell M$ is free of rank r over $R/\ell R$.

We claim that A is smooth. This is automatic if $\text{char } k = 0$. So suppose that $\text{char } k = p > 0$. By [25, Theorem 4.2], we have $p \nmid c$, so by the above, M/pM is free of rank r

over R/pR . Applying $\mathrm{Hom}_R(M, -)$ to

$$0 \rightarrow \mathrm{Lie} E \rightarrow E(k[\epsilon]/(\epsilon^2)) \rightarrow E(k) \rightarrow 0$$

yields

$$0 \rightarrow \mathrm{Hom}_R(M, \mathrm{Lie} E) \rightarrow A(k[\epsilon]/(\epsilon^2)) \rightarrow A(k) \rightarrow 0.$$

Thus

$$\mathrm{Lie} A \simeq \mathrm{Hom}_R(M, \mathrm{Lie} E) \simeq \mathrm{Hom}_{R/pR}(M/pM, \mathrm{Lie} E) \simeq (\mathrm{Lie} E)^r.$$

In particular, $\dim \mathrm{Lie} A = r$, so A is smooth.

Since A is also proper, it is an extension of a finite étale commutative group scheme Φ by an abelian variety B . The constructed surjection $A \rightarrow E^r$ with finite kernel restricts to a homomorphism $B \rightarrow E^r$ with finite kernel, and it must still be surjective since E^r does not have algebraic subgroups of finite index; thus B is isogenous to E^r . Since $B(\bar{k})$ is divisible, the extension splits over \bar{k} . In particular, for each prime ℓ ,

$$\#A(\bar{k})[\ell] = \#E(\bar{k})[\ell]^r \# \Phi[\ell]. \quad (6)$$

On the other hand, by definition of A ,

$$A(\bar{k})[\ell] = \mathrm{Hom}_R(M, E(\bar{k})[\ell]) = \mathrm{Hom}_{R/\ell R}(M/\ell M, E(\bar{k})[\ell]). \quad (7)$$

We claim that

$$\#A(\bar{k})[\ell] = \#E(\bar{k})[\ell]^r. \quad (8)$$

If $\ell \nmid c$, then $M/\ell M$ is free of rank r over $R/\ell R$, so (8) holds; in particular, this holds if $\ell = \mathrm{char} k$. Now suppose that $\ell | c$. Then $R/\ell R \simeq \mathbb{F}_\ell[e]/(e^2)$. Every module over $R/\ell R$ is a direct sum of copies of \mathbb{F}_ℓ and $\mathbb{F}_\ell[e]/(e^2)$. Since R is saturated in $\mathrm{End} E$, the homomorphisms

$$\frac{R}{\ell R} \rightarrow \frac{\mathrm{End} E}{\ell(\mathrm{End} E)} \rightarrow \mathrm{End} E(\bar{k})[\ell]$$

are injective, but $\#E(\bar{k})[\ell] = \ell^2 = \#(R/\ell R)$, so $E(\bar{k})[\ell]$ is free of rank 1 over $R/\ell R$. The equality $\# \mathrm{Hom}_{R/\ell R}(N, R/\ell R) = \#N$ holds for $N = \mathbb{F}_\ell$ and $N = \mathbb{F}_\ell[e]/(e^2)$, so it holds for every finite $(R/\ell R)$ -module N , and in particular for $M/\ell M$. Thus (7) implies

$$\#A(\bar{k})[\ell] = \#(M/\ell M) = \#(R/\ell R)^r = \#E(\bar{k})[\ell]^r.$$

Thus (8) holds for all ℓ .

Comparing (6) and (8) shows that $\#\Phi[\ell] = 1$ for all ℓ , so Φ is trivial. Thus $A = B$, an abelian variety.

- (b) By Lemma 4.3 below, it suffices to show that if $M \rightarrow P$ is an injection of modules with P projective, then $\mathcal{H}om_R(P, E) \rightarrow \mathcal{H}om_R(M, E)$ is surjective. We have an exact sequence

$$0 \rightarrow \mathcal{H}om_R(P/M, E) \rightarrow \mathcal{H}om_R(P, E) \rightarrow \mathcal{H}om_R(M, E).$$

By (a), $\mathcal{H}om_R(P, E)$ and $\mathcal{H}om_R(M, E)$ are abelian varieties, so the image I of $\mathcal{H}om_R(P, E) \rightarrow \mathcal{H}om_R(M, E)$ is an abelian subvariety of $\mathcal{H}om_R(M, E)$. By Proposition 4.1,

$$\dim \mathcal{H}om_R(P, E) = \dim \mathcal{H}om_R(P/M, E) + \dim \mathcal{H}om_R(M, E),$$

so $\dim I = \dim \mathcal{H}om_R(M, E)$. Thus $I = \mathcal{H}om_R(M, E)$; i.e., $\mathcal{H}om_R(P, E) \rightarrow \mathcal{H}om_R(M, E)$ is surjective.

- (c) Since $\mathcal{H}om_R(-, E)$ is exact, it transforms the co-image of g into the image of f . (Co-image equals image in any abelian category, though the proof above does not need this.)

- (d) The proof of [25, Proposition A.2] shows that $\mathcal{H}om_R(R/I, E) \simeq E[I]$ (there R is *equal* to $\text{End } E$, but this is not used). The proof of [25, Proposition A.3] shows that $E/E[I]$ is the connected component of $\mathcal{H}om_R(I, E)$, but $\mathcal{H}om_R(I, E)$ is already connected, by (a).
- (e) The function $(\#T)^{2/\text{rk } R}$ of T is multiplicative in short exact sequences. So is $\# \mathcal{H}om_R(T, E)$, since $\mathcal{H}om_R(-, E)$ is exact. Thus we may reduce to the case in which T is simple, i.e., $T \simeq R/I$ for some maximal ideal I . Then $\mathcal{H}om_R(T, E) = E[I]$ by (d). We have $I \supseteq \ell R$ for some prime ℓ . If $I = \ell R$, then $E[I] = E[\ell]$, which has order $\ell^2 = \#(R/I)^{2/\text{rk } R}$.

Now suppose that $I \neq \ell R$. If R has rank 2, then $\#(R/I) = \ell$; if R has rank 4, then $\#(R/I) = \ell^2$. Choose $f \in I \setminus \ell R$; then f does not kill $E[\ell]$, so $E[I] \subsetneq E[\ell]$. Thus $\#E[I] \leq \ell = \#(R/I)^{2/\text{rk } R}$. Thus $\# \mathcal{H}om_R(T, E) \leq (\#T)^{2/\text{rk } R}$ holds for each Jordan–Hölder factor of $R/\ell R$, but for $T = R/\ell R$ equality holds, so all the inequalities must have been equalities.

- (f) Start with the exact sequence

$$0 \rightarrow E[n] \rightarrow E \xrightarrow{n} E.$$

Given $S \in \mathcal{C}$, apply the left exact functors $\text{Hom}_{\mathcal{C}}(S, -)$ and then $\text{Hom}_R(M, -)$; taken for all S , this produces an exact sequence of representable functors

$$0 \rightarrow \mathcal{H}om_R(M, E[n]) \rightarrow \mathcal{H}om_R(M, E) \xrightarrow{n} \mathcal{H}om_R(M, E).$$

Hence $\mathcal{H}om_R(M, E[n]) \simeq A[n]$.

- (g) We have

$$\begin{aligned} T_\ell A &:= \varprojlim_e A[\ell^e](k_s) \\ &\simeq \varprojlim_e \mathcal{H}om_R(M, E[\ell^e])(k_s) && \text{(by (f))} \\ &:= \varprojlim_e \text{Hom}_R(M, E[\ell^e](k_s)) \\ &\simeq \text{Hom}_R(M, \varprojlim_e E[\ell^e](k_s)) \\ &=: \text{Hom}_R(M, T_\ell E). \end{aligned} \quad \square$$

The following was used in the proof of Theorem 4.2(b).

Lemma 4.3. *Let \mathcal{C} be an abelian category with enough projectives. Let $F: \mathcal{C}^{\text{opp}} \rightarrow \mathcal{D}$ be a left exact functor. Suppose that for each monomorphism $M \rightarrow P$ with P projective, the morphism $FP \rightarrow FM$ is an epimorphism. Then F is exact.*

Proof. Given $A \in \mathcal{C}$, choose an epimorphism $P \rightarrow A$ with P projective, and let K be the kernel. The sequence $0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$ yields

$$0 \rightarrow FA \rightarrow FP \rightarrow FK \rightarrow (R^1 F)A \rightarrow (R^1 F)P = 0,$$

and the hypothesis implies that $FP \rightarrow FK$ is surjective, so $(R^1 F)A = 0$. This holds for all A , so F is exact. \square

Remark 4.4. The hypothesis that R is saturated in Theorem 4.2 cannot be dropped. For example, if E is an elliptic curve over \mathbb{C} with $\text{End } E = \mathbb{Z}[i]$, and R is the subring $\mathbb{Z}[2i]$, then

the R -module $\mathbb{Z}[i]$ has a presentation

$$R \begin{pmatrix} 2i \\ -2 \end{pmatrix} \xrightarrow{\quad} R^2 \begin{pmatrix} 1 & i \end{pmatrix} \xrightarrow{\quad} \mathbb{Z}[i] \longrightarrow 0,$$

so by definition,

$$\mathcal{H}om_R(\mathbb{Z}[i], E) \simeq \ker \left(E^2 \begin{pmatrix} 2i & -2 \end{pmatrix} \xrightarrow{\quad} E \right) \simeq E \times E[2],$$

which is not an abelian variety. Moreover, applying $\mathcal{H}om_R(-, E)$ to the injection $\mathbb{Z}[i] \xrightarrow{2} R$ yields a homomorphism $E \rightarrow E \times E[2]$, which is not surjective, so $\mathcal{H}om_R(-, E)$ is not exact. Finally, $\mathbb{Z}[i]$ is isomorphic as R -module to the R -ideal $I := 2\mathbb{Z}[i]$, and $\mathcal{H}om_R(I, E)$ is not an abelian variety.

4.3. Duality of abelian varieties. Let E be an elliptic curve over a field k . Let $R := \text{End } E$. The *Rosati involution*, sending an endomorphism to its dual, is an isomorphism $R \rightarrow R^{\text{opp}}$. If M is a left R -module, then $M^* := \text{Hom}_R(M, R)$ (the group of homomorphisms of left R -modules) is a *right* R -module: given $f \in M^*$ and $r \in R$, let $f \cdot r$ be the composition $M \xrightarrow{f} R \xrightarrow{r} R$. In other words, M^* is a left R^{opp} -module, which we may view as a left R -module by using the Rosati involution. Moreover, if M is f.p., then it is finite over \mathbb{Z} , and then so is M^* . Also, M^* is torsion-free.

Given an abelian variety A , let A^\vee be the dual abelian variety. The following lets us understand the duals of abelian varieties arising from modules.

Theorem 4.5. *Given a f.p. torsion-free left R -module M , we have*

$$\mathcal{H}om_R(M, E)^\vee \simeq \mathcal{H}om_R(M^*, E),$$

functorially in M .

Proof. Let M be a f.p. torsion-free left R -module. Choose a presentation

$$R^n \xrightarrow{P} R^m \longrightarrow M \longrightarrow 0, \tag{9}$$

where $P \in M_{m \times n}(R)$. Apply $\mathcal{H}om_R(-, E)$ to obtain

$$0 \longrightarrow A \longrightarrow E^m \xrightarrow{P^T} E^n$$

where P^T is the transpose of P . Taking dual abelian varieties yields

$$E^n \xrightarrow{P^\dagger} E^m \longrightarrow A^\vee \longrightarrow 0, \tag{10}$$

where P^\dagger is obtained from P by applying the Rosati involution entrywise.

On the other hand, applying $\text{Hom}_R(-, R)$ to (9) yields

$$0 \longrightarrow M^* \longrightarrow R^m \xrightarrow{P^{T\dagger}} R^n$$

and applying $\mathcal{H}om_R(-, E)$ yields

$$E^n \xrightarrow{P^\dagger} E^m \longrightarrow \mathcal{H}om_R(M^*, E) \longrightarrow 0.$$

Comparing with (10) shows that

$$\mathcal{H}om_R(M^*, E) \simeq A^\vee = \mathcal{H}om_R(M, E)^\vee. \tag{11}$$

Given a homomorphism of f.p. torsion-free left R -modules $M \xrightarrow{f} N$, we can build a commutative diagram

$$\begin{array}{ccccccc} R^n & \longrightarrow & R^m & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow f & & \\ R^i & \longrightarrow & R^j & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

and apply the constructions above to show that (11) is functorial in M . \square

4.4. The other Hom functor. Under the assumptions of Theorem 4.2(b), we have a functor of additive categories

$$\begin{aligned} \mathcal{H}om_R(-, E) : \{\text{f.p. torsion-free left } R\text{-modules}\}^{\text{opp}} \\ \rightarrow \{\text{abelian varieties isogenous to a power of } E\}, \end{aligned}$$

as promised in the introduction.

Given an abelian variety A over the same field as E , the abelian group $\text{Hom}(A, E)$ (the group of homomorphisms of abelian varieties) is a left $(\text{End } E)$ -module, and hence also a left R -module, and it is f.p. because it is f.p. over \mathbb{Z} [15, p. 178, Corollary 1]. In fact, we get a functor in the opposite direction:

$$\begin{aligned} \text{Hom}(-, E) : \{\text{abelian varieties isogenous to a power of } E\} \\ \rightarrow \{\text{f.p. torsion-free left } R\text{-modules}\}^{\text{opp}}. \end{aligned}$$

For which elliptic curves E are $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ inverse equivalences of categories? If we start with the R -module R and apply $\mathcal{H}om_R(-, E)$ and then $\text{Hom}(-, E)$, we obtain $\text{End } E$. Thus we assume from now on that $R = \text{End } E$.

Theorem 4.6. *Let E be an elliptic curve over a field. Let $R := \text{End } E$. Then*

- (a) *The functor $\mathcal{H}om_R(-, E)$ is fully faithful.*
- (b) *The functor $\text{Hom}(-, E)$ on the image of $\mathcal{H}om_R(-, E)$ is an inverse to $\mathcal{H}om_R(-, E)$.*
- (c) *The image of $\mathcal{H}om_R(-, E)$ consists exactly of the products of elliptic curves of the form $\mathcal{H}om_R(I, E)$ for a nonzero left R -ideal I .*

Proof.

- (a) The ring R is \mathbb{Z} , a quadratic order, or a maximal quaternionic order. By Theorem 3.1, Theorem 3.2(i), or Theorem 3.3(b), respectively, every f.p. torsion-free left R -module is a finite direct sum of nonzero left R -ideals. Thus, (a) follows if for any two nonzero R -ideals I and J , the natural map

$$\text{Hom}_R(J, I) \longrightarrow \text{Hom}(\mathcal{H}om_R(I, E), \mathcal{H}om_R(J, E))$$

is an isomorphism. If $R = \mathbb{Z}$, this is trivial. If R is a quadratic order, this is the elliptic curve case of the isomorphism given in (48) in [9, Proposition 17]. If R is a maximal quaternionic order, then by Theorem 3.3(a) all f.p. torsion-free left R -modules are projective, i.e., direct summands of f.p. free left R -modules; since $\mathcal{H}om_R(-, E)$ is fully faithful when restricted to free modules, it is also fully faithful on projective modules.

- (b) This is a general property of fully faithful functors.
- (c) As remarked in the proof of (a), every f.p. torsion-free left R -module is a finite direct sum of nonzero left R -ideals I . \square

5. MAXIMAL ABELIAN VARIETIES OVER \mathbb{F}_{p^2}

Fix a prime p . Call an abelian variety A over \mathbb{F}_{p^2} *maximal* if A has the maximum possible number of \mathbb{F}_{p^2} -points for its dimension, namely $(p+1)^{2\dim A}$.

Proposition 5.1. *Let A be a g -dimensional abelian variety over \mathbb{F}_{p^2} . Let ℓ be a prime not equal to p . The following are equivalent:*

- (a) *The abelian variety A is maximal; i.e., $\#A(\mathbb{F}_{p^2}) = (p+1)^{2g}$.*
- (b) *The characteristic polynomial of π_A on $T_\ell A$ equals $(x+p)^{2g}$.*
- (c) *We have $\pi_A = -p$.*
- (d) *We have $A(\mathbb{F}_{p^2}) \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^{2g}$ as abelian groups.*

Proof.

- (a) \Rightarrow (b): Let $\lambda_1, \dots, \lambda_{2g} \in \overline{\mathbb{Q}}$ be the eigenvalues of π_A acting on $T_\ell A$. Then $|\lambda_i| = p$ and $\#A(\mathbb{F}_{p^2}) = \prod (1 - \lambda_i) = \prod |1 - \lambda_i| \leq (p+1)^{2g}$; if equality holds, then $\lambda_i = -p$ for all i . Thus the characteristic polynomial is $(x+p)^{2g}$.
- (b) \Rightarrow (c): Since π_A is determined by its action on $T_\ell A$, which is semisimple [15, pp. 203–206], we obtain $\pi_A = -p$.
- (c) \Rightarrow (d): We have

$$A(\mathbb{F}_{p^2}) = \ker(\pi_A - 1) = \ker(-p - 1) = A[p+1] \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^{2g}.$$

- (d) \Rightarrow (a): Trivial. □

Lemma 5.2. *If A and B are maximal abelian varieties over \mathbb{F}_{p^2} , then any homomorphism $A_{\overline{\mathbb{F}}_p} \rightarrow B_{\overline{\mathbb{F}}_p}$ is the base extension of a homomorphism $A \rightarrow B$.*

Proof. Any homomorphism respects the p^2 -power Frobenius endomorphisms (both are equal to $-p$), and hence descends to \mathbb{F}_{p^2} . □

Every supersingular elliptic curve over $\overline{\mathbb{F}}_p$ admits a unique model over \mathbb{F}_{p^2} that is maximal: the existence is [1, Lemma 3.21], and uniqueness follows from Lemma 5.2. In particular, maximal elliptic curves over \mathbb{F}_{p^2} exist. If E is any such curve, then E is supersingular, and Lemma 5.2 implies that $\text{End } E = \text{End } E_{\overline{\mathbb{F}}_p}$, which is a maximal order \mathcal{O} in a quaternion algebra over \mathbb{Q} ramified at p and ∞ . Also, the kernel of the p -power Frobenius morphism $E \rightarrow E^{(p)}$ is isomorphic to α_p .

By Honda–Tate theory, any maximal abelian variety A over \mathbb{F}_{p^2} is isogenous to a power of E . The main result of this section strengthens this as follows:

Theorem 5.3.

- (a) *Every maximal abelian variety A over \mathbb{F}_{p^2} is isomorphic to a product of maximal elliptic curves over \mathbb{F}_{p^2} .*
- (b) *Fix a maximal elliptic curve E over \mathbb{F}_{p^2} . Let $\mathcal{O} := \text{End } E$. Then the functors $\mathcal{H}\text{om}_{\mathcal{O}}(-, E)$ and $\text{Hom}(-, E)$ are inverse equivalences of categories. Also, the categories involved can be rewritten so that $\mathcal{H}\text{om}_{\mathcal{O}}(-, E)$ becomes*

$$\begin{aligned} \mathcal{H}\text{om}_{\mathcal{O}}(-, E) &: \{\text{f.p. projective left } \mathcal{O}\text{-modules}\}^{\text{opp}} \\ &\xrightarrow{\sim} \{\text{maximal abelian varieties}/\mathbb{F}_{p^2}\}. \end{aligned}$$

(c) Fix a maximal elliptic curve E over \mathbb{F}_{p^2} . Let $g \geq 2$. Every g -dimensional maximal abelian variety over \mathbb{F}_{p^2} is isomorphic to E^g . In particular, any product of g maximal elliptic curves over \mathbb{F}_{p^2} is isomorphic to any other.

The analogous results hold if maximal is replaced by minimal; i.e., we consider abelian varieties A over \mathbb{F}_{p^2} such that $\#A(\mathbb{F}_{p^2}) = (p-1)^{2\dim A}$.

We need a few lemmas for the proof of Theorem 5.3.

Lemma 5.4. *There exists an elliptic curve E over \mathbb{F}_p such that $E_{\mathbb{F}_{p^2}}$ is maximal.*

Proof. By Honda–Tate theory, there exists an elliptic curve E over \mathbb{F}_p with $p+1$ points. The p -power Frobenius endomorphism π_E of E satisfies $\pi_E^2 = -p$, so $E_{\mathbb{F}_{p^2}}$ satisfies condition (c) in Proposition 5.1. \square

Lemma 5.5. *If E and E' are maximal elliptic curves over \mathbb{F}_{p^2} , there exists a separable isogeny $E \rightarrow E'$.*

Proof. By transitivity, we may assume that E is the elliptic curve of Lemma 5.4. Since $\#E(\mathbb{F}_{p^2}) = \#E'(\mathbb{F}_{p^2})$, there exists an isogeny $E \rightarrow E'$, which factors as $E \xrightarrow{f} E \xrightarrow{\lambda} E'$, where f is a power of the p -power Frobenius morphism (which goes from E to itself since E is definable over \mathbb{F}_p), and λ is separable. \square

Remark 5.6. Even better, if E and E' are maximal elliptic curves over \mathbb{F}_{p^2} , there exists an isogeny of ℓ -power degree for any prime $\ell \neq p$: for an argument due to Serre, see [14, p. 223].

Lemma 5.7. *If A is a maximal abelian variety over \mathbb{F}_{p^2} , then every finite étale subgroup scheme of $A_{\overline{\mathbb{F}}_p}$ is defined over \mathbb{F}_{p^2} .*

Proof. The p^2 -power Frobenius field automorphism acts on prime-to- p torsion points of $A_{\overline{\mathbb{F}}_p}$ as $-p$, so it preserves any finite subgroup of order prime to p . \square

Lemma 5.8. *Let A be a supersingular abelian variety over a field k of characteristic p . Every p -power order subgroup scheme $G \subseteq A$ is an iterated extension of copies of α_p .*

Proof. By induction, it suffices to show that if $G \neq 0$, then G contains a copy of α_p . The a -number $\dim_k \mathrm{Hom}(\alpha_p, G)$ is unchanged by field extension [13, Section 1.5], so we may assume that k is algebraically closed. Then A is isogenous to a power E^r of a supersingular elliptic curve. The group scheme $E[p]$ is an extension of α_p by α_p , so all Jordan–Hölder factors of $E[p^n]$ are isomorphic to α_p . The image of $E[p^N]$ under the isogeny $E^r \rightarrow A$ contains $A[p^n]$ if N is sufficiently large relative n , and $A[p^n]$ contains G if n is large enough. Thus all Jordan–Hölder factors of G are isomorphic to α_p . \square

Lemma 5.9. *Let E and E' be maximal elliptic curves over \mathbb{F}_{p^2} . Identify α_p with a subgroup scheme of each. Then each homomorphism $E \rightarrow E'$ restricts to a homomorphism $\alpha_p \rightarrow \alpha_p$ and the resulting map*

$$\mathrm{Hom}(E, E') \rightarrow \mathrm{End} \alpha_p \simeq \mathbb{F}_{p^2} \quad (12)$$

is surjective.

Proof. Since each α_p is the kernel of the p -power Frobenius morphism, any homomorphism $E \rightarrow E'$ must map α_p to α_p . If $E' = E$, then the resulting ring homomorphism

$$\mathrm{End} E \rightarrow \mathrm{End} \alpha_p \simeq \mathbb{F}_{p^2}$$

is surjective because every ring homomorphism from \mathcal{O} to \mathbb{F}_{p^2} is surjective. In the general case, Lemma 5.5 provides a separable isogeny $\lambda: E \rightarrow E'$; then $\lambda|_{\alpha_p} \neq 0$, so $\{\lambda \circ e : e \in \text{End } E\}$ surjects onto $\text{End } \alpha_p$. \square

Lemma 5.10. *Let B be a product of g maximal elliptic curves over \mathbb{F}_{p^2} . Then $\text{Aut } B$ acts transitively on the set of subgroup schemes of B isomorphic to α_p . Also, if ℓ is a prime not equal to p , then $\text{Aut } B$ acts transitively on the set of subgroup schemes of B of order ℓ .*

Proof. For $\ell \neq p$, Tate's theorem on homomorphisms [24] shows that

$$\text{End } B \rightarrow \text{End}(T_\ell B) \simeq M_{2g}(\mathbb{Z}_\ell)$$

is an isomorphism. In particular, it surjects onto $\text{End } B[\ell] \simeq M_{2g}(\mathbb{F}_\ell)$, and $\text{Aut } B \simeq \text{GL}_{2g}(\mathbb{Z}_\ell)$ surjects on $\text{Aut } B[\ell] \simeq \text{GL}_{2g}(\mathbb{F}_\ell)$. For any finite-dimensional vector space V , the group $\text{GL}(V)$ acts transitively on the lines in V , so $\text{Aut } B$ acts transitively on the order ℓ subgroup schemes of B .

If $F_p: B \rightarrow B^{(p)}$ is the p -power Frobenius morphism, then $\ker F_p \simeq \alpha_p^g$. Lemma 5.9 implies that the ring homomorphism

$$\text{End } B \rightarrow \text{End}(\ker F_p) = \text{End}(\alpha_p^g) = M_g(\mathbb{F}_{p^2})$$

is surjective, so $\text{Aut } B \rightarrow \text{GL}_g(\mathbb{F}_{p^2})$ is surjective. The latter group acts transitively on the copies of α_p in α_p^g over \mathbb{F}_{p^2} . \square

Corollary 5.11. *Let B be a product $E_1 \times \cdots \times E_g$ of maximal elliptic curves over \mathbb{F}_{p^2} . Let H be a subgroup scheme of B such that $H \simeq \alpha_p$ or $\#H$ is a prime $\ell \neq p$. Then B/H is a product of maximal elliptic curves over \mathbb{F}_{p^2} .*

Proof. By Lemma 5.10, we may assume that H is contained in the copy of α_p in E_1 , or a cyclic subgroup of order ℓ contained in E_1 . Then E_1/H is another maximal elliptic curve over \mathbb{F}_{p^2} , and $B/H \simeq (E_1/H) \times E_2 \times \cdots \times E_g$. \square

Proof of Theorem 5.3.

- (a) Among all isogenies from a product of maximal elliptic curves to A , let $\phi: B \rightarrow A$ be one of minimal degree (at least one such ϕ exists, by Tate's theorem on homomorphisms [24]). Let G be the connected component of $\ker \phi$.

Suppose that ϕ is inseparable. Then $G \neq 0$. By Lemma 5.8, G contains a copy of α_p . By Corollary 5.11, B/α_p is again a product of maximal elliptic curves. Now ϕ factors as $B \rightarrow B/\alpha_p \rightarrow A$, and $B/\alpha_p \rightarrow A$ contradicts the minimality of ϕ .

Similarly, if ϕ is separable and $\deg \phi > 1$, then $\ker \phi$ contains a subgroup H of order ℓ , defined over \mathbb{F}_{p^2} by Lemma 5.7; Corollary 5.11 shows again that $B/H \rightarrow A$ contradicts the minimality of ϕ .

Hence ϕ is an isomorphism, so A is a product of maximal elliptic curves.

- (b) First let us justify the rewriting of the categories. F.p. torsion-free left \mathcal{O} -modules are projective by Theorem 3.3(a). By Proposition 5.1 and Honda–Tate theory, the abelian varieties isogenous to a power of E are exactly the maximal abelian varieties over \mathbb{F}_{p^2} .

By (a), every maximal abelian variety is a product of maximal elliptic curves, each of which is $\mathcal{H}om_{\mathcal{O}}(I, E)$ for some left \mathcal{O} -ideal I , by the bottom of page 541 in [25]. The result now follows from 4.6.

- (c) Combine Theorem 3.3(c) and part (b).

The same proofs apply in the minimal case. \square

Remark 5.12. Because of Lemma 5.2, Theorem 5.3(c) could be deduced also from its analogue over $\overline{\mathbb{F}}_p$, that for $g \geq 2$, any product of g supersingular elliptic curves over $\overline{\mathbb{F}}_p$ is isomorphic to any other. The latter is a well-known theorem of Deligne, proved in a similar way: see [16, Theorem 6.2] and [22, Theorem 3.5].

Remark 5.13. A related result can be found in [17]: Theorem 2 there states that if A is an abelian variety over an algebraically closed field of characteristic p , and the a -number of A equals $\dim A$, then A is isomorphic to a product of supersingular elliptic curves.

6. KERNEL SUBGROUPS

6.1. General properties of kernel subgroups.

Definition 6.1. Let A be an abelian variety over a field. Call a subgroup scheme $G \subseteq A$ a *kernel subgroup* if $G = A[I]$ for some $I \subseteq \text{End } A$. (These are called *ideal subgroups* in [9, p. 302].)

In the definition, we may replace I by the left $(\text{End } A)$ -ideal it generates without changing $A[I]$. Thus we may always assume that I is a left $(\text{End } A)$ -ideal.

Proposition 6.2.

- (a) *An intersection of kernel subgroups in A is a kernel subgroup.*
- (b) *Let A_1, \dots, A_n be abelian varieties. Suppose that $G_i \subseteq A_i$ for $i = 1, \dots, n$. Then $\prod_{i=1}^n G_i$ is a kernel subgroup of $\prod_{i=1}^n A_i$ if and only if each G_i is a kernel subgroup of A_i .*
- (c) *Let I_1, \dots, I_n be pairwise coprime 2-sided ideals of $\text{End } A$. Let $G_i \subseteq A[I_i]$ for $i = 1, \dots, n$. Then $\sum_{i=1}^n G_i$ is a kernel subgroup if and only if each G_i is a kernel subgroup.*

Proof.

- (a) We have $\bigcap A[I_i] = A[\sum I_i]$ for any left ideals I_i .
- (b) Let $A = \prod A_i$ and $G = \prod G_i$. Suppose that $G = A[I]$. For each $f \in I$, the composition $A_i \hookrightarrow A \xrightarrow{f} A \twoheadrightarrow A_i$ defines $\bar{f} \in \text{End } A_i$, and G_i is the intersection of the kernels of all such \bar{f} .

Conversely, suppose that $G_i = A_i[I_i]$ for each i . Let $I := \prod I_i$ denote the set of “diagonal” endomorphisms $(f_1, \dots, f_n): A \rightarrow A$ with $f_i \in I_i$. Then $G = A[I]$.

- (c) By induction, we may assume $n = 2$. Since I_1 and I_2 are coprime 2-sided ideals, $A[I_1 I_2] = A[I_1] \oplus A[I_2]$ and every subgroup scheme $H \subseteq A[I_1 I_2]$ decomposes as $H_1 \oplus H_2$ where $H_i \subseteq A[I_i]$; namely $H_i = H \cap A[I_i]$.

If $G_1 + G_2$ is a kernel subgroup, then so is $G_i = (G_1 + G_2) \cap A[I_i]$, by (a).

Conversely, suppose that $G_i = A[J_i]$ for some left ideal J_i . Replace J_i by $J_i + I_i$ to assume that $J_i \supseteq I_i$. Let $K := I_2 J_1 + I_1 J_2$. We claim that $A[K] = G_1 + G_2$. First, $I_2 J_1 \subseteq J_1$, which kills G_1 ; also, $I_1 J_2 \subseteq I_1$, which kills G_1 . Thus K kills G_1 . Similarly, K kills G_2 . Thus $G_1 + G_2 \subseteq A[K]$. On the other hand, if we write $A[K] = H_1 \oplus H_2$ with $H_i \subseteq A[I_i]$, we will show that $H_i \subseteq G_i$, so that $A[K] \subseteq G_1 + G_2$. Write $1 = e_1 + e_2$ with $e_i \in I_i$. Then the subsets $e_1 J_1 \subseteq I_2 J_1 \subseteq K$ and $e_2 J_1 \subseteq I_2$ kill H_1 , so J_1 kills H_1 ; i.e., $H_1 \subseteq A[J_1] = G_1$. Similarly $H_2 \subseteq G_2$. So $A[K] \subseteq G_1 + G_2$. Hence $G_1 + G_2 = A[K]$, a kernel subgroup. \square

6.2. Kernel subgroups of a power of an elliptic curve.

Proposition 6.3. *Let E be an elliptic curve over a field, and let $r \in \mathbb{Z}_{\geq 0}$. Let $R := \text{End } E$. For a subgroup scheme $G \subseteq E^r$, the following are equivalent:*

- (i) G is a kernel subgroup.
- (ii) G is the kernel of a homomorphism $E^r \rightarrow E^s$ for some $s \in \mathbb{Z}_{\geq 0}$.
- (iii) There exists a f.p. torsion-free R -module M such that $E^r/G \simeq \mathcal{H}om_R(M, E)$.
- (iv) There exists a submodule $M \subseteq R^r$ such that applying $\mathcal{H}om_R(-, E)$ to

$$0 \rightarrow M \rightarrow R^r \rightarrow R^r/M \rightarrow 0$$

yields

$$0 \rightarrow G \rightarrow E^r \rightarrow E^r/G \rightarrow 0.$$

Proof.

(i) \Rightarrow (ii): Suppose that G is a kernel subgroup, say $A[I]$. Let f_1, \dots, f_n be generators for

I . Then G is the kernel of $E^r \xrightarrow{(f_1, \dots, f_n)} (E^r)^n$.

(ii) \Rightarrow (iii): This is a special case of Theorem 4.2(c).

(iii) \Rightarrow (iv): If $E^r/G \simeq \mathcal{H}om_R(M, E)$ for some f.p. torsion-free M , then by Theorem 4.6(a), the natural surjection $E^r \twoheadrightarrow E^r/G$ comes from some injection $M \hookrightarrow R^r$. Applying $\mathcal{H}om_R(-, E)$ to

$$0 \rightarrow M \rightarrow R^r \rightarrow R^r/M \rightarrow 0$$

yields

$$0 \rightarrow H \rightarrow E^r \twoheadrightarrow E^r/G$$

for some H , which must be isomorphic to G .

(iv) \Rightarrow (ii): Choose a surjection $h: R^s \twoheadrightarrow M$. Applying $\mathcal{H}om_R(-, E)$ to the composition $R^s \xrightarrow{h} M \hookrightarrow R^r$ produces a homomorphism $E^r \twoheadrightarrow E^r/G \hookrightarrow E^s$ with kernel G .

(ii) \Rightarrow (i): We may increase s to assume that $r|s$. Then G is an intersection of s/r endomorphisms of E^r , so it is a kernel subgroup by Proposition 6.2(a). \square

Proposition 6.4. *Let E be an elliptic curve. Let $R := \text{End } E$. Then the following are equivalent:*

- (i) For each $r \in \mathbb{Z}_{\geq 0}$, every subgroup scheme of E^r is a kernel subgroup.
- (ii) For each $r \in \mathbb{Z}_{\geq 0}$, every finite subgroup scheme of E^r is a kernel subgroup.
- (iii) The functors $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ are inverse equivalences of categories.

Proof.

(i) \Rightarrow (ii): Trivial.

(ii) \Rightarrow (iii): Suppose that A is an abelian variety isogenous to E^r . Then $A \simeq E^r/G$ for some finite subgroup scheme G . By assumption, G is a kernel subgroup. Proposition 6.3(i) \Rightarrow (iii) implies that A is in the image of $\mathcal{H}om_R(-, E)$. The result now follows from Theorem 4.6.

(iii) \Rightarrow (i): Let G be a subgroup scheme of E^r . Then E^r/G is isogenous to E^s for some $s \leq r$. By assumption, $\mathcal{H}om_R(-, E)$ is an equivalence of categories, so E^r/G is of the form $\mathcal{H}om_R(M, E)$. By Proposition 6.3(iii) \Rightarrow (i), G is a kernel subgroup. \square

In the next few sections, we investigate when it holds that all finite subgroup schemes of powers of E are kernel subgroups, in order to determine when $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ are inverse equivalences of categories.

6.3. Prime-to- p subgroups. We continue to assume that E is an elliptic curve and $R = \text{End } E$. Let ℓ be a prime not equal to $\text{char } k$. Let $R_\ell := R \otimes \mathbb{Z}_\ell$. The natural map $R_\ell \rightarrow \text{End}_{\mathbb{Z}_\ell} T_\ell E$ is injective since an endomorphism that kills $E[\ell^n]$ for all n is 0, and has saturated image since an endomorphism that kills $E[\ell]$ is equal to ℓ times an endomorphism. Let $C := \text{End}_{R_\ell} T_\ell E$, which is the commutant of R_ℓ in $\text{End}_{\mathbb{Z}_\ell} T_\ell E \simeq M_2(\mathbb{Z}_\ell)$. For any elliptic curve, we have $\text{rk } R \in \{1, 2, 4\}$, so one of the following holds:

- (i) $R_\ell = \mathbb{Z}_\ell$ and $C = M_2(\mathbb{Z}_\ell)$;
- (ii) $R_\ell = C = \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \alpha$, a \mathbb{Z}_ℓ -algebra that is a saturated rank 2 \mathbb{Z}_ℓ -submodule of $M_2(\mathbb{Z}_\ell)$ for some $\alpha \in M_2(\mathbb{Z}_\ell)$; or
- (iii) $R_\ell = M_2(\mathbb{Z}_\ell)$ and $C = \mathbb{Z}_\ell$.

(To see that $C = R_\ell$ in case (ii), one may argue as follows. By [15, Corollary 3 in III.§19], the \mathbb{Q} -algebra $R \otimes \mathbb{Q}$ is semisimple, so $R \otimes \mathbb{Q}_\ell$ is either a degree 2 field extension of \mathbb{Q}_ℓ , or is conjugate to $\mathbb{Q}_\ell \times \mathbb{Q}_\ell$. In either case, the commutant $C \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ of $R \otimes \mathbb{Q}_\ell$ in $M_2(\mathbb{Q}_\ell)$ is 2-dimensional. On the other hand, an algebra generated by one element is commutative, so C contains R_ℓ . Also, R_ℓ is saturated in $M_2(\mathbb{Z}_\ell)$. The previous three sentences imply that $C = R_\ell$.)

Let $e \in \mathbb{Z}_{>0}$.

Lemma 6.5. *Every finitely generated left $C/\ell^e C$ -module injects into a free $C/\ell^e C$ -module.*

Proof. If $C = \mathbb{Z}_\ell$, this is trivial. The statement that every finitely generated module injects into a finitely generated projective module is preserved when a ring R is replaced $M_n(R)$ for any $n \geq 1$; thus the case $C = M_2(\mathbb{Z}_\ell)$ follows.

Finally, suppose that C is of rank 2. Then $C/\ell^e C$ is free of rank 2 over $\mathbb{Z}/\ell^e \mathbb{Z}$; say with basis 1, α . For $c \in C/\ell^e C$, let $\lambda(c)$ be the coefficient of α in c . Multiplying any nonzero element of $\ker \lambda$ by α gives an element outside $\ker \lambda$. Therefore the pairing

$$\begin{aligned} C/\ell^e C \times C/\ell^e C &\longrightarrow \mathbb{Z}/\ell^e \mathbb{Z} \\ x, y &\longmapsto \lambda(xy) \end{aligned}$$

is a perfect pairing. In other words, the Pontryagin dual $(C/\ell^e C)^D$ is isomorphic to $C/\ell^e C$ as a $C/\ell^e C$ -module. If M is a finitely generated $C/\ell^e C$ -module, there exists a surjection $(C/\ell^e C)^r \twoheadrightarrow M^D$ for some $r \in \mathbb{Z}_{\geq 0}$; taking Pontryagin duals yields an injection $M \hookrightarrow ((C/\ell^e C)^r)^D \simeq (C/\ell^e C)^r$. \square

Lemma 6.6. *The group $(T_\ell E)^2$ is free as an R_ℓ -module and as a C -module. The group $E[\ell^e](k_s)^2$ is free as an $R/\ell^e R$ -module and as a $C/\ell^e C$ -module.*

Proof. Since $E[\ell^e](k_s) = T_\ell E/\ell^e T_\ell E$, by Nakayama's lemma it is enough to check that $E[\ell](k_s)^2$ is free as an A -module, for $A = R/\ell R$ and for $A = C/\ell C$. Identify $E[\ell](k_s)^2$ with \mathbb{F}_ℓ^2 , so that $A \subseteq M_2(\mathbb{F}_\ell)$. The case $A = \mathbb{F}_\ell$ is trivial. If A is $\mathbb{F}_\ell \oplus \mathbb{F}_\ell \alpha$ for some $\alpha \in M_2(\mathbb{F}_\ell)$, then every faithful A -module of dimension 2 over \mathbb{F}_ℓ is free. If $A = M_2(\mathbb{F}_\ell)$, then the free A -module A is a direct sum of two copies of \mathbb{F}_ℓ^2 (the two column spaces). \square

Lemma 6.7. *The natural maps*

$$\begin{aligned} C/\ell^e C &\rightarrow \text{End}_{R/\ell^e R} E[\ell^e](k_s) \\ R/\ell^e R &\rightarrow \text{End}_{C/\ell^e C} E[\ell^e](k_s) \end{aligned}$$

are isomorphisms.

Proof. The first map is an isomorphism since $C = \text{End}_{R_\ell} T_\ell E$ and C and R_ℓ are saturated in $\text{End } T_\ell E \simeq M_2(\mathbb{Z}_\ell)$. Lemma 6.6 implies that $E[\ell^e](k_s)$ is a progenerator of the category of finitely generated $R/\ell^e R$ -modules, so Morita duality yields the second isomorphism. \square

Recall that $\mathcal{G}_k = \text{Gal}(k_s/k)$. There is a group homomorphism $\mathcal{G}_k \rightarrow C^\times$ since each $\sigma \in \mathcal{G}_k$ respects the R -action on the groups $E[\ell^e](k_s)$ and $T_\ell E$.

Proposition 6.8. *Let E be an elliptic curve over a field k . Let ℓ, e, C be as above. Let G be a subgroup scheme of $E[\ell^e]^r$ for some r . Then G is a kernel subgroup if and only if $G(k_s)$ is a $C/\ell^e C$ -submodule of $E[\ell^e]^r(k_s)$.*

Proof. Suppose that $G(k_s)$ is a $C/\ell^e C$ -submodule of $E[\ell^e]^r(k_s)$. Let $H := E[\ell^e]^r/G$. Then $H(k_s)$ is a finitely generated $C/\ell^e C$ -module. By Lemma 6.5, $H(k_s)$ injects into a free $C/\ell^e C$ -module, which in turn injects into $E[\ell^e]^s(k_s)$ for some s . Because of the homomorphisms $\mathcal{G}_k \rightarrow C^\times \rightarrow (C/\ell^e C)^\times$, the $C/\ell^e C$ -module homomorphism $H(k_s) \rightarrow E[\ell^e]^s(k_s)$ is a \mathcal{G}_k -module homomorphism, so it comes from a homomorphism $H \rightarrow E[\ell^e]^s$ of étale group schemes. The composition $E[\ell^e]^r \rightarrow H \hookrightarrow E[\ell^e]^s$ is given by an $s \times r$ matrix N_e with entries in $\text{End}_{C/\ell^e C} E[\ell^e](k_s) = R/\ell^e R$ (the equality is Lemma 6.7). Lift N_e to $N \in M_{s \times r}(R)$. Then G is the intersection of the kernel subgroups $E[\ell^e]^r$ and $\ker(N: E^r \rightarrow E^s)$. By Propositions 6.2(a) and 6.3 G is a kernel subgroup.

Conversely, if G is a kernel subgroup, say the kernel of $E^r \rightarrow E^s$, then it is also the kernel of $E[\ell^e]^r \rightarrow E[\ell^e]^s$, which is a homomorphism of $C/\ell^e C$ -modules, so G is a $C/\ell^e C$ -module. \square

The group homomorphism $\mathcal{G}_k \rightarrow C^\times$ induces algebra homomorphisms $\mathbb{Z}_\ell[\mathcal{G}_k] \rightarrow C$ and $\mathbb{F}_\ell[\mathcal{G}_k] \rightarrow C/\ell C$.

Proposition 6.9. *Let E, k, ℓ, R, R_ℓ, C be as above. The following are equivalent:*

- (i) *The homomorphism $\mathbb{F}_\ell[\mathcal{G}_k] \rightarrow C/\ell C$ is surjective.*
- (ii) *The homomorphism $\mathbb{Z}_\ell[\mathcal{G}_k] \rightarrow C$ is surjective.*
- (iii) *Every ℓ -power order subgroup scheme of E^r for every r is a kernel subgroup.*

Proof.

(i) \Rightarrow (ii): Nakayama's lemma.

(ii) \Rightarrow (iii): Let G be an ℓ -power subgroup scheme of E^r , say $G \subseteq E[\ell^e]^r$. Then $G(k_s)$ is a $\mathbb{Z}_\ell[\mathcal{G}_k]$ -module. Since $\mathbb{Z}_\ell[\mathcal{G}_k] \rightarrow C$ is surjective, $G(k_s)$ is also a C -module, and hence a $C/\ell^e C$ -module. By Proposition 6.8, G is a kernel subgroup.

(iii) \Rightarrow (i): Suppose that $\mathbb{F}_\ell[\mathcal{G}_k] \rightarrow C/\ell C$ is not surjective; let D be the image. The algebra $C/\ell C$ is one of \mathbb{F}_ℓ , $\{(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix})\} \simeq \mathbb{F}_\ell[\epsilon]/(\epsilon^2)$, $\{(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix})\} \simeq \mathbb{F}_\ell \times \mathbb{F}_\ell$, \mathbb{F}_{ℓ^2} , or $M_2(\mathbb{F}_\ell)$. The first is excluded since it has no nontrivial subalgebras. In the second, third, and fourth cases, D can only be \mathbb{F}_ℓ , and it is easy to find a subspace of $\mathbb{F}_\ell^2 \simeq E[\ell](k_s)$ that is not a $C/\ell C$ -module. In the fifth case, D is contained in a copy of either $\{(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix})\}$ or \mathbb{F}_{ℓ^2} . Now $\{(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix})\}$ fixes a line in \mathbb{F}_ℓ^2 not fixed by $M_2(\mathbb{F}_\ell)$. And \mathbb{F}_{ℓ^2} fixes an \mathbb{F}_{ℓ^2} -line in $\mathbb{F}_{\ell^2}^2 \simeq E^2[\ell](k_s)$ that is not fixed by $M_2(\mathbb{F}_\ell)$. Thus in each case, there is a subgroup scheme of $E[\ell]$ or $E^2[\ell]$ that is not a $C/\ell C$ -module, and hence by Proposition 6.8 not a kernel subgroup. \square

6.4. p -power subgroups.

Proposition 6.10. *Let E be an ordinary elliptic curve over a field k of characteristic p . Assume that $\text{End } E \neq \mathbb{Z}$ (automatic if k is finite). Then every p -power order subgroup scheme $G \subseteq E^r$ is a kernel subgroup.*

Proof. The ring $R := \text{End } E \simeq \text{End } E_{\bar{k}}$ is a quadratic order. Although R is not necessarily a Dedekind domain, its conductor is prime to p , so it makes sense to speak of the splitting behavior of (p) in R . In fact, since E is ordinary, (p) splits, say as $\mathfrak{p}\mathfrak{q}$. So $E[p]$ is the direct sum of group schemes $E[\mathfrak{p}]$ and $E[\mathfrak{q}]$, each of order p by Theorem 4.2(e). Since E is ordinary, one of them, say $E[\mathfrak{p}]$, is étale, and the other is connected. For any $e \in \mathbb{Z}_{\geq 0}$, we have $(p^e) = \mathfrak{p}^n \mathfrak{q}^n$ so $E[p^e] \simeq E[\mathfrak{p}^e] \oplus E[\mathfrak{q}^e]$. The Jordan–Hölder factors of $E[\mathfrak{p}^e]$ are isomorphic to $E[\mathfrak{p}]$, so $E[\mathfrak{p}^e]$ is étale; similarly $E[\mathfrak{q}^e]$ is connected. We have $G \subseteq E[p^e]^r$ for some e . By Proposition 6.2(c), we may assume that $G \subseteq E[\mathfrak{p}^e]^r$ or $G \subseteq E[\mathfrak{q}^e]^r$.

In the first case, $E[\mathfrak{p}^e](k^s) \simeq \mathbb{Z}/p^e\mathbb{Z}$, so G is the kernel of a homomorphism $E[\mathfrak{p}^e]^r \rightarrow E[\mathfrak{p}^e]^s$ given by a matrix in $M_{s \times r}(\mathbb{Z})$. Since $E[\mathfrak{p}^e]$ is a kernel subgroup, so is $E[\mathfrak{p}^e]^r$, and so is G , by Propositions 6.2(a) and 6.3.

In the second case, we take Cartier duals: $E[\mathfrak{p}^e]^r \twoheadrightarrow G^\vee$. Then G^\vee is the cokernel of some homomorphism $E[\mathfrak{p}^e]^s \rightarrow E[\mathfrak{p}^e]^r$ given by a matrix $N \in M_{r \times s}(\mathbb{Z})$. So G is the kernel of the homomorphism $E[\mathfrak{q}^e]^r \rightarrow E[\mathfrak{q}^e]^s$ given by the transpose $N^T \in M_{s \times r}(\mathbb{Z})$. Since $E[\mathfrak{q}^e]$ is a kernel subgroup, so is $E[\mathfrak{q}^e]^r$, and so is G , by Propositions 6.2(a) and 6.3. \square

Proposition 6.11. *Let E be a supersingular elliptic curve over a field k of characteristic p .*

- (a) *If $k = \mathbb{F}_p$, then every p -power order subgroup scheme $G \subseteq E^r$ is a kernel subgroup, and in fact is a kernel of an endomorphism of E^r .*
- (b) *If $k = \mathbb{F}_{p^2}$ and $\text{rk } R = 4$ (i.e., $\#E(\mathbb{F}_{p^2}) = (p \pm 1)^2$), then every subgroup scheme $G \subseteq E^r$ is a kernel subgroup.*
- (c) *If $k = \mathbb{F}_{p^2}$ and $\text{rk } R \neq 4$, then there exists a copy of α_p in $E \times E$ that is not a kernel subgroup.*
- (d) *If k is \mathbb{F}_{p^a} for some $a \geq 3$, or if k is infinite, then there exists a copy of α_p in $E \times E$ that is not a kernel subgroup.*

Proof. The kernel of $\pi_{E,p}: E \rightarrow E^{(p)}$ is α_p . Suppose that $\alpha_p \subseteq E$ is a kernel subgroup. By Proposition 6.3(i) \Rightarrow (iv), $\alpha_p \simeq \mathcal{H}\text{om}_R(R/I, E)$ for some left R -ideal I . By Theorem 4.2(e), $p = \#(R/I)^{2/\text{rk } R}$. We have three cases:

- If $R = \mathbb{Z}$, this is a contradiction.
- If $\text{rk } R = 2$, then $\#(R/I) = p$, so $R/I \simeq \mathbb{F}_p$. Since E is supersingular, p is ramified or inert in R , and the above implies that p is ramified.
- If $\text{rk } R = 4$, then $\#(R/I) = p^2$, so I is the unique ideal of index p^2 in R , and $R/I \simeq \mathbb{F}_{p^2}$.

If J is an R -module with $I^2 \subsetneq J \subsetneq R^2$ (here I^2 means $I \times I$), then $R^2/J \simeq R/I$ (since R/I is a field), and the surjection $R^2 \twoheadrightarrow R^2/J$ gives rise to an injection $\alpha_p \hookrightarrow E \times E$. Conversely, any kernel subgroup $\alpha_p \subseteq E \times E$ arises from such a J . So such kernel subgroups are in bijection with $\mathbb{P}^1(R/I)$. On the other hand, $\text{End } \alpha_p \simeq k$, so $\text{Hom}(\alpha_p, E \times E) = k^2$, and the copies of α_p in $E \times E$ are in bijection with $\mathbb{P}^1(k)$. Thus if every α_p in $E \times E$ is a kernel subgroup, then $\mathbb{P}^1(R/I)$ is in bijection with $\mathbb{P}^1(k)$, so $\#(R/I) = \#k$; i.e., $k \simeq R/I$, which is \mathbb{F}_p or \mathbb{F}_{p^2} as above. This proves (c) and (d).

- (a) By Lemma 5.8, $E^r \rightarrow E^r/G$ factors as a chain of p -isogenies, each with kernel α_p . If we show that any quotient E^r/α_p is isomorphic to E^r , then each abelian variety in the chain must be isomorphic to E^r , so G is a kernel of an endomorphism of E^r , as desired.

- The group $\mathrm{GL}_r(\mathbb{Z}) \subseteq \mathrm{GL}_r(\mathrm{End} E)$ acts on E^r , and acts transitively on the nonzero elements of $\mathrm{Hom}(\alpha_p, E^r) = \mathbb{F}_p^r$. Therefore it suffices to consider the quotient E^r/α_p in which the α_p is contained in $E \times 0 \times \cdots \times 0$. Now $E/\alpha_p = E/E[\pi_E] \simeq E$, so $E^r/\alpha_p \simeq E^r$.
- (b) The abelian variety E^r/G is isogenous to a power of E , so by Theorem 5.3(b), it is of the form $\mathcal{H}om_R(M, E)$. By Proposition 6.3(iii) \Rightarrow (i), G is a kernel subgroup. \square

7. ABELIAN VARIETIES ISOGENOUS TO A POWER OF AN ELLIPTIC CURVE

Let E be an elliptic curve over k . We break into cases, first according to whether E is ordinary or supersingular, and next according to $\mathrm{rk} \mathrm{End} E$ and $\#k$. By convention, elliptic curves over a field of characteristic 0 are included among the ordinary curves.

7.1. E is ordinary and $\mathrm{rk} \mathrm{End} E = 1$.

Theorem 7.1. *Fix an elliptic curve E over a field k such that $\mathrm{End} E \simeq \mathbb{Z}$.*

- (a) *The image of $\mathcal{H}om_R(-, E)$ consists of abelian varieties isomorphic to a power of E .*
- (b) *The functors $\mathcal{H}om_R(-, E)$ and $\mathrm{Hom}(-, E)$ are inverse equivalences of categories (i.e., every abelian variety isogenous to a power of E is isomorphic to a power of E) if and only if $\mathrm{char} k = 0$ and for every prime ℓ the homomorphism $\mathbb{F}_\ell[\mathcal{G}_k] \rightarrow \mathrm{End} E[\ell](k_s) \simeq M_2(\mathbb{F}_\ell)$ is surjective.*

Proof of Theorem 7.1.

- (a) Every f.p. torsion-free \mathbb{Z} -module is free.
- (b) By Proposition 6.4, $\mathcal{H}om_R(-, E)$ and $\mathrm{Hom}(-, E)$ are equivalences if and only if every finite subgroup scheme G is a kernel subgroup. By Proposition 6.2(c), we need only consider G of prime power order.

If $\mathrm{char} k = p > 0$, then $\# \ker \pi_{E,p} = p$, but $\#E[I]$ is a square for every nonzero ideal $I \subseteq \mathbb{Z}$, so $\ker \pi_{E,p}$ is not a kernel subgroup. If $\mathrm{char} k = 0$, then apply Proposition 6.9(i) \Leftrightarrow (iii) for every ℓ . \square

7.2. E is ordinary and $\mathrm{rk} \mathrm{End} E = 2$. Fix an ordinary elliptic curve E over a field k such that $\mathrm{rk} \mathrm{End} E = 2$. (These are called *CM elliptic curves* in [9, Section 3].) Then $\mathrm{End} E \simeq \mathrm{End} E_{\bar{k}}$, because if an endomorphism becomes divisible by a positive integer n over an extension field, it kills $E[n]$, so it is divisible by n already over k . Let $R := \mathrm{End} E$ and $K := \mathrm{Frac} R$.

If E' is an elliptic curve isogenous to E , then $\mathrm{End} E'$ is another order R' in K . Let $f_{E'}$ be the conductor of R' , i.e., the index of R' in its integral closure. More generally, if A is an abelian variety isogenous to E^r , then $\mathrm{End} A$ is an order in $M_r(K)$, and its center $Z(\mathrm{End} A)$ is an order in $Z(M_r(K)) = K$, and we let f_A be the conductor of $Z(\mathrm{End} A)$.

Theorem 7.2. *Fix an ordinary elliptic curve E over a field k such that $\mathrm{rk} \mathrm{End} E = 2$. Let $R := \mathrm{End} E$. The image of $\mathcal{H}om_R(-, E)$ consists of the abelian varieties A isogenous to a power of E such that $f_A | f_E$, i.e., such that $R \subseteq Z(\mathrm{End} A)$. These are exactly the products of elliptic curves E' each isogenous to E and satisfying $f_{E'} | f_E$.*

Proof. Suppose that $\phi: E^r \rightarrow A$ is an isogeny and $f_A | f_E$. Since $f_A | f_E$, there is an R -action on A such that ϕ respects the R -actions. Let $G := \ker \phi$, so $G(k_s)$ is an R -module. Write $G = \bigoplus_\ell G_\ell$, where G_ℓ is a group scheme of ℓ -power order. For $\ell \neq \mathrm{char} k$, we are in the case $R_\ell = C$ of Section 6.3, so $G_\ell(k_s)$ is also a $C/\ell^e C$ -module for some e , and Proposition 6.8

shows that G_ℓ is a kernel subgroup. If $\text{char } k = p > 0$, then G_p is a kernel subgroup by Proposition 6.10. By Proposition 6.2(c), G is a kernel subgroup. By Proposition 6.3(i) \Rightarrow (iii), the abelian variety $A \simeq E^r/G$ is in the image of $\mathcal{H}om_R(-, E)$.

Conversely, if A is in the image of $\mathcal{H}om_R(-, E)$ then by Theorem 4.6(c), A is a product of elliptic curves of the form $\mathcal{H}om_R(I, E)$. Because the functor $\mathcal{H}om_R(-, E)$ is fully faithful, if $E' = \mathcal{H}om_R(I, E)$ then E' is isogenous to E and $\text{End } E' \simeq \text{End}_R I$, which contains R since R is commutative. In particular, $f_{E'}|f_E$. Finally, f_A is the least common multiple of the $f_{E'}$, so $f_A|f_E$ too. \square

Theorem 7.3. *Fix an ordinary elliptic curve E over a finite field \mathbb{F}_q . Let $R := \text{End } E$. Then $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ are equivalences of categories if and only if $\mathbb{Z}[\pi_E] = R$.*

Proof. Suppose that $\mathbb{Z}[\pi_E] = R$. If A is isogenous to a power of E , then π_A has the same minimal polynomial as π_E , so $Z(\text{End } A)$ contains $\mathbb{Z}[\pi_A] \simeq \mathbb{Z}[\pi_E]$; i.e., $f_A|f_E$ is automatic.

On the other hand, if $\mathbb{Z}[\pi_E] \neq R$, then E is isogenous to an elliptic curve E' satisfying $\text{End } E' = \mathbb{Z}[\pi_{E'}]$ [25, Theorem 4.2(2)]. Theorem 7.2 shows that E' is not in the image of $\mathcal{H}om_R(-, E)$, so $\mathcal{H}om_R(-, E)$ is not an equivalence of categories. \square

We can also give a criterion that applies even if k is not finite.

Theorem 7.4. *Fix an ordinary elliptic curve E over a field k such that $\text{rk } \text{End } E = 2$. Then $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ are equivalences of categories if and only if for every prime $\ell \neq \text{char } k$, there exists $\sigma \in \mathcal{G}_k$ whose action on $E[\ell](k_s)$ is not multiplication by a scalar.*

Proof. By Propositions 6.4(i) \Leftrightarrow (iii), 6.2(c), and 6.10, the functors $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ are equivalences if and only if for each $\ell \neq \text{char } k$, the homomorphism $\mathbb{F}_\ell[\mathcal{G}_k] \rightarrow C/\ell C$ is surjective. Since $\dim_{\mathbb{F}_\ell} C/\ell C = 2$, surjectivity is equivalent to the image of $\mathbb{F}_\ell[\mathcal{G}_k] \rightarrow C/\ell C \subseteq \text{End } E[\ell](k_s) \simeq M_2(\mathbb{F}_\ell)$ being \mathbb{F}_ℓ . \square

Remark 7.5. If E and E' are ordinary elliptic curves over an algebraically closed field k and their endomorphism rings are orders in the same quadratic field, then E and E' are isogenous. But over non-algebraically closed fields, this can fail. For example, if E is an ordinary elliptic curve over a finite field, then its quadratic twist E' has the same endomorphism ring, but opposite trace of Frobenius, so E and E' are not isogenous.

7.3. E is supersingular and $k = \mathbb{F}_p$. Fix a supersingular elliptic curve E over \mathbb{F}_p . Let $R := \text{End } E$. Let $P(x)$ be the characteristic polynomial of $\pi := \pi_E$. Define f_A as in Section 7.2. In particular, f_E is the conductor of R . We have the following cases:

prime	$P(x)$	$\mathbb{Z}[\pi]$	$R = \text{End } E$	f_E	equivalence?
$p \not\equiv 3 \pmod{4}$	$x^2 + p$	$\mathbb{Z}[\sqrt{-p}]$	$\mathbb{Z}[\sqrt{-p}]$	1	YES
$p \equiv 3 \pmod{4}$	$x^2 + p$	$\mathbb{Z}[\sqrt{-p}]$	$\mathbb{Z}[\sqrt{-p}]$	2	YES
$p \equiv 3 \pmod{4}$	$x^2 + p$	$\mathbb{Z}[\sqrt{-p}]$	$\mathbb{Z}\left[\frac{1+\sqrt{-p}}{2}\right]$	1	NO
$p = 2$	$x^2 \pm 2x + 2$	$\mathbb{Z}[i]$	$\mathbb{Z}[i]$	1	YES
$p = 3$	$x^2 \pm 3x + 3$	$\mathbb{Z}\left[\frac{-1+\sqrt{-3}}{2}\right]$	$\mathbb{Z}\left[\frac{-1+\sqrt{-3}}{2}\right]$	1	YES

The last column, which indicates when $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ are equivalences of categories, is explained by the following analogues of Theorems 7.2 and 7.3, proved in the same way except that we use Proposition 6.11(a) in place of Proposition 6.10.

Theorem 7.6. Fix a supersingular elliptic curve E over \mathbb{F}_p . Let $R := \text{End } E$. The image of $\mathcal{H}om_R(-, E)$ consists of the abelian varieties A isogenous to a power of E such that $f_A|f_E$, i.e., such that $R \subseteq Z(\text{End } A)$. These are exactly the products of elliptic curves E' each isogenous to E and satisfying $f_{E'}|f_E$.

Theorem 7.7. Fix a supersingular elliptic curve E over \mathbb{F}_p . Let $R := \text{End } E$. Then $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ are equivalences of categories if and only if $\mathbb{Z}[\pi_E] = R$.

7.4. E is supersingular, $k = \mathbb{F}_{p^2}$, and $\text{rk End } E = 4$. In this case, E is a maximal or minimal elliptic curve over \mathbb{F}_{p^2} . These cases were already handled: see Theorem 5.3.

7.5. E is supersingular, $k = \mathbb{F}_{p^2}$, and $\text{rk End } E = 2$. By Proposition 6.11(c), not every subgroup scheme is a kernel subgroup. By Proposition 6.4(iii) \Leftrightarrow (ii), the functors $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ are not equivalences of categories.

Remark 7.8. These are the cases in which the characteristic polynomial of π_E is one of $x^2 + px + p^2$, $x^2 + p^2$, or $x^2 - px + p^2$. Hence $\pi_E = p\zeta$ for a root of unity ζ of order 3, 4, or 6, respectively. But p does not divide the conductor of R , so $\zeta \in R$. Now $\zeta \in \text{Aut } E$, so E has j -invariant 0 or 1728.

7.6. E is supersingular and $\#k > p^2$. By Proposition 6.11(d), not every subgroup scheme is a kernel subgroup. By Proposition 6.4(iii) \Leftrightarrow (ii), the functors $\mathcal{H}om_R(-, E)$ and $\text{Hom}(-, E)$ are not equivalences of categories.

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